More about matrices & linear functions.

<u>Calc 1:</u> $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\Longrightarrow f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

$$\underbrace{f(x_0+h) - f(x_0)}_{\text{change in } f} \approx f'(x_0)h$$

$$\text{change in } f \approx f'(x_0)(\text{change in } x)$$

Thinking of matrices as functions.

<u>example:</u> $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

We can think of this as a linear function from $\mathbb{R}^2$ to $\mathbb{R}^2$

$F: \mathbb{R}^2 \to \mathbb{R}^2$

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ -x+y \end{pmatrix}$$

What the derivative matrix tells us:

If $G : \mathbb{R}^k \to \mathbb{R}^n$

Then $\underset{\text{in } \mathbb{R}^k}{G'(x_0)}$ is a $n \times k$ matrix.

and $G(x_0+h) - G(x_0) \approx \underset{n\times k \;\; k\times 1}{G'(x_0)\,h} = \begin{array}{c} n\times 1 \\ \text{vector.} \end{array}$

$$x_0 = \begin{pmatrix} x_{0,1} \\ \vdots \\ x_{0,k} \end{pmatrix} \qquad h = \begin{pmatrix} h_1 \\ \vdots \\ h_k \end{pmatrix}$$

Thus $G(x_0+h) \approx \underbrace{G(x_0) + G'(x_0)h}_{}$

good 1st order approximation of the function.

<u>Example</u>

$$\text{let } U(x,y) = \begin{pmatrix} x^2 - 3xy \\ \cos(y-2x) \end{pmatrix}$$

Let's calculate the derivative matrix at $(1,2)$ and figure out what it tells us.

$$U(1,2) = \begin{pmatrix} 1^2 - 3(1)(2) \\ \cos(2-2(1)) \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$$

$$U'(x,y) = \begin{pmatrix} u_{1x} & u_{1y} \\ u_{2x} & u_{2y} \end{pmatrix}$$

$$= \begin{pmatrix} 2x-3y & -3x \\ 2\sin(y-2x) & -\sin(y-2x) \end{pmatrix}$$

$$\overset{\text{at } (1,2)}{=} \begin{pmatrix} 2(1)-3(2) & -3(1) \\ 2\sin(2-2(1)) & -\sin(2-(2)(1)) \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -3 \\ 0 & 0 \end{pmatrix} = U'(1,2)$$

This tells us that

$$U((1,2)+(h_1,h_2)) \simeq U(1,2) + U'(1,2)\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$\Rightarrow U(1+h_1, 2+h_2) \simeq \begin{pmatrix} -5 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 & -3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$U(1+h_1, 2+h_2) \simeq \begin{pmatrix} -5 - 4h_1 - 3h_2 \\ 1 \end{pmatrix}$$

Best linear approximation to $U(x,y)$ near $(1,2)$

<u>Example</u> Let $B(x,y,z) = x^2 - 3y - xe^{2yz}$

Find the derivative matrix at $(2,0,3)$. What does this tell us?

$B: \mathbb{R}^3 \to \mathbb{R}^1$

$B(2,0,3) = 4 - 0 - 2e^0 = 2$

$\Rightarrow B'$ is a 1×3 matrix

$B' = \nabla B$

$B' = \nabla B = (B_x, B_y, B_z)$ at $(2,0,3)$

$= \left(2x - e^{2yz}, -3 - 2xze^{2yz}, -2xye^{2yz}\right)$

$= (4-1, -3-12, 0) = (3, -15, 0)$

$B(x_0 + h) - B(x_0) \approx B'(x_0)h$

$B\left((2,0,3) + (h_1, h_2, h_3)\right) - B(2,0,3) \approx (3, -15, 0)\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$

$B(2+h_1, h_2, 3+h_3) - 2 \approx 3h_1 - 15h_2$

Let's check with some numbers.

$h_1 = .1$, $h_2 = .1$, $h_3 = .05$

$B(2.1, .1, 3.05) - 2 \approx 3(.1) - 15(.1)$

≈ -1.2

$$B(x,y,z) = x^2 - 3y - xe^{2yz}$$

$$= 2.1^2 - 3(.1) - 2.1\, e^{2(.1)(3.05)} \approx 2 + -1.2$$

$$\approx 0.8 \quad 0.245$$

In general, if $F : \mathbb{R}^k \to \mathbb{R}^1$

Then $F(x_0 + h) - F(x_0) \approx \nabla F \cdot h$

$$F(x_0 + h) \approx F(x_0) + \nabla F \cdot h$$

$$F(x_0 + h) \approx F(x_0) + F_{x_1}(x_0) h_1 + F_{x_2}(x_0) h_2 + \dots + F_{x_k}(x_0) h_k$$

Best linear approximation of the fcn at x_0 $\longrightarrow$ gives the equation of the tangent plane, if you graph it.

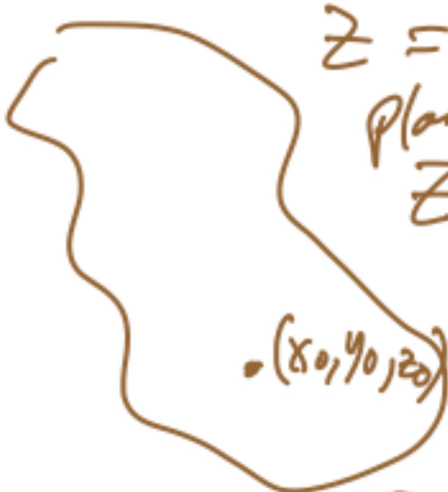
Example $F : \mathbb{R}^2 \to \mathbb{R}$.

$$F((x_0, y_0) + (h_1, h_2)) \approx F(x_0, y_0) + F_x(x_0, y_0) h_1$$

$$+ F_y(x_0,y_0)\, h_2$$

Graph of function

$$z = F(x,y), \quad h_1 = x - x_0, \quad h_2 = y - y_0$$



plane

$$z = F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)$$

$$z_0 = F(x_0,y_0)$$

also known as the first order Taylor polynomial at (x_0,y_0)

Chain Rule for fcns of several variables

Case 1: $y = f(x) \quad x = g(t)$

$$(f \circ g)(t) = f(g(t))$$

$$[f \circ g]'(t) = f'(g(t)) \cdot g'(t)$$

example: $\left(e^{\tan(\theta)}\right)' = e^{\tan\theta} \cdot \sec^2\theta$

Several variables:

$$F:\mathbb{R}^k\to\mathbb{R}^n \quad G:\mathbb{R}^n\to\mathbb{R}^p$$

For $x\in\mathbb{R}^k$, $G(F(x))\in\mathbb{R}^p$

Let $H=G\circ F:\mathbb{R}^k\to\mathbb{R}^p$ be defined by $H(x)=G(F(x))$, then

$$H'(x)=G'(F(x))\cdot F'(x)$$

matrix mult.

Example: Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=x^2y-2y^3$

let $\alpha(t)=(\cos(t),2\sin(t))$

ellipse.

Let $g(t)=f(\alpha(t))$

Find $g'(\frac{\pi}{4})$.

① Compute composition first.

② Use Chain Rule.

$$f(x,y)=x^2y-2y^3$$

① $g(t) = f(\check{\alpha}(t)) = f(\cos(t), 2\sin(t))$

$= \cos^2(t)(2\sin(t)) - 2(2\sin(t))^3$

$= 2\cos^2(t)\sin(t) - 16\sin^3(t)$

$g'(t) = 4\cos(t)(-\sin t)\sin(t) + 2\cos^2(t)\cos(t)$

$- 16(3)\sin^2(t)\cdot\cos(t)$

$= -52\sin^2(t)\cos(t) + 2\cos^3(t)$

$t = \pi/4$

$g'(t) = -52\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}\right)^3$

$= (-52+2)\frac{1}{2\sqrt{2}} = \boxed{-\frac{25}{\sqrt{2}}}$

② Chain rule:

$g'(t) = f'(\alpha(t))\,\alpha'(t).$ $\quad t = \frac{\pi}{4}$

$\alpha(t) = (\cos(t), 2\sin(t))$

$\alpha'(t) = (-\sin(t), 2\cos(t))$ $\quad = \left(\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$

$= \left(-\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ $\quad = \left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$

$f' = \nabla f = \nabla(x^2y - 2y^3)$

$= (2xy, x^2 - 6y^2)$

plug in $\alpha(t)$

$= \left(2\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}), \frac{1}{2} - 6(2)\right)$.

$$= \left(2, \frac{-23}{2}\right)$$

Chain Rule

$$g'(t) = f'(\alpha(t))\alpha'(t)$$

$$= \left(2, \frac{-23}{2}\right)\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix}$$

$$= -\frac{2}{\sqrt{2}} + -\frac{23\sqrt{2}}{2} = \frac{-2}{\sqrt{2}} + \frac{-23}{\sqrt{2}}$$

$$= \boxed{\frac{-25}{\sqrt{2}}}$$

Suppose we look at the surface (manifold) $f(x, y, z, \dots) = C$

$f: \mathbb{R}^n \to \mathbb{R}$

Then $f(\alpha(t)) = C$

Take derivative of both sides.

$$f'(\alpha(t))\cdot\alpha'(t) = 0$$

$$\nabla f(\alpha(t)) \cdot \underbrace{\alpha'(t)}_{\text{tangent vector}} = 0$$

$\Rightarrow$ The ∇f is perpendicular to every tangent vector.

ie ∇f is $\perp$ to level sets of f.